Final Exam (online section). Name: _____

- *Twenty-five questions, equal weight.*
- *Due in dropbox by 11:59 pm*
- *Be sure to show all relevant work!*

#1. Given $f(x) = x^2 - 2x$, evaluate and simplify: $f(x) - 2f(-x)$

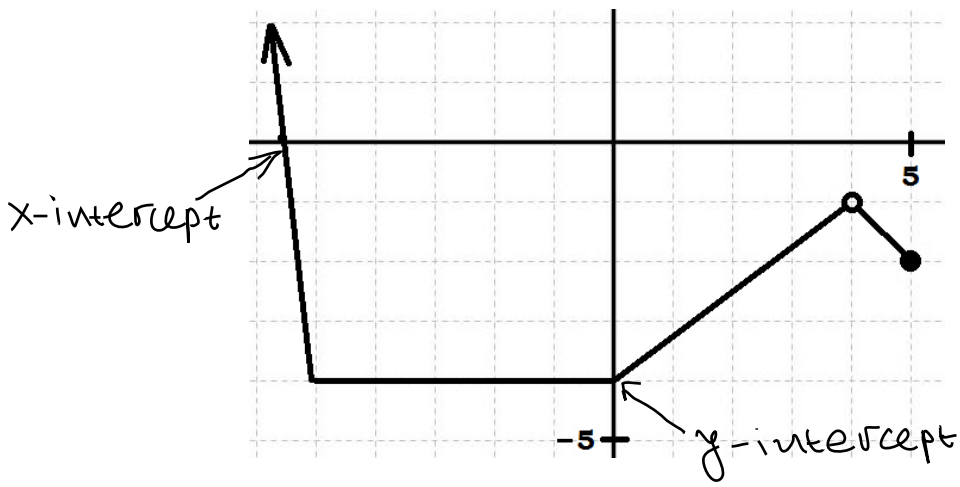
$f(-x) = (-x)^2 - 2(-x) = x^2 + 2x$

$f(x) - 2f(-x) = x^2 - 2x - 2(x^2 + 2x) =$

$= x^2 - 2x - 2x^2 - 4x =$

$= -x^2 - 6x$

#2. Behold the graph of a function:



a) Identify any x-intercept(s) $x = -5, 5$

b) Identify any y-intercept $y = -4$

c) State the domain $(-\infty, 4) \cup (4, 5)$

d) State the range $[-4, \infty)$

e) State any interval(s) over which the function is increasing $(0, 4)$

f) State any interval(s) over which the function is decreasing $(-\infty, -5) \cup (4, 5)$

g) State any interval(s) over which the function is constant $(-5, 0)$

h) This function has an absolute minimum of -4 at $-5 \leq x \leq 0$.

#3. Here's another nice function: $f(x) = \dfrac{3x^2 - 4}{x^2 - 4}$

a) Identify any vertical asymptote(s) $x^2 - 4 = 0 \rightarrow x = \pm 2$

b) Identify any horizontal asymptote(s) $\lim_{x \to \pm\infty} \dfrac{3x^2 - 4}{x^2 - 4} = 3$

c) Is this function even, odd, both, or neither? $f(-x) = \dfrac{3(-x)^2 - 4}{(-x)^2 - 4} = \dfrac{3x^2 - 4}{x^2 - 4} = f(x)$ even

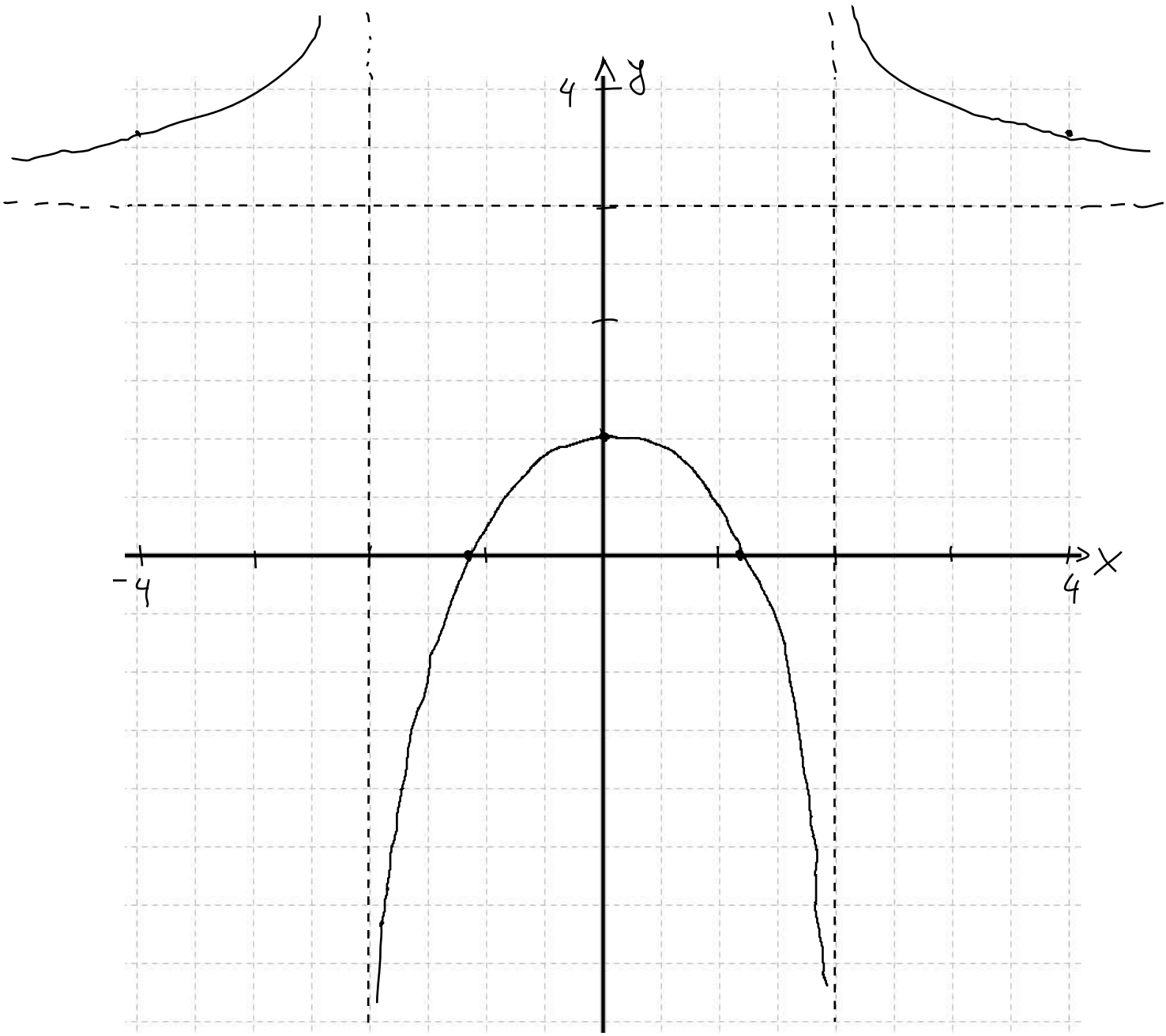
d) Locate any x-intercept(s). $3x^2 - 4 = 0 \rightarrow x = \pm \dfrac{2}{\sqrt{3}} = \pm 1.2$

e) Locate any y-intercept(s). $f(0) = \dfrac{-4}{-4} = 1$

f) Plot all of the above information on the grid below.

g) Evaluate f(4) and plot the point. $f(4) = \dfrac{3(4^2) - 4}{4^2 - 4} = \dfrac{48 - 4}{16 - 4} = \dfrac{44}{12} = \dfrac{11}{3} = 3.7$

h) Using only the above information, finish the (approximate) graph of f.



#4. Penelope's Pistachio Emporium sells pistachios by the jar. If you buy a dozen or fewer jars, the charge is \$8.00 per jar; additional jars are \$7.25 each. Shipping is a flat charge of \$4.50 on all orders. Write a piecewise-linear model for the cost of x jars.

Dozen or fewer: $0 \le x \le 12$

Cost $c(x) = 4.50 + 8x$

More than dozen; $x > 12$

\$7.25/jar

cost $C(x) = 4.50 + (8)(12) + 7.25(x-12) =$

\$8 for first dozen jars

$= 4.50 + 96 + 7.25x - 87$

$= 13.5 + 7.25x$

Summary:

$$C(x) = \begin{cases} 4.50 + 8x, & 0 \le x \le 12 \\ 13.5 + 7.25x, & x > 12 \end{cases}$$

#5. Start with the parent function f(x) = |x|.

a) If f(x) is shifted right by 5 to form g(x), then g(x) is horizontally compressed by a factor of 1/2 to form h(x), then what is h's equation?

$g(x) = |x-5|$

$h(x) = |2x-5|$

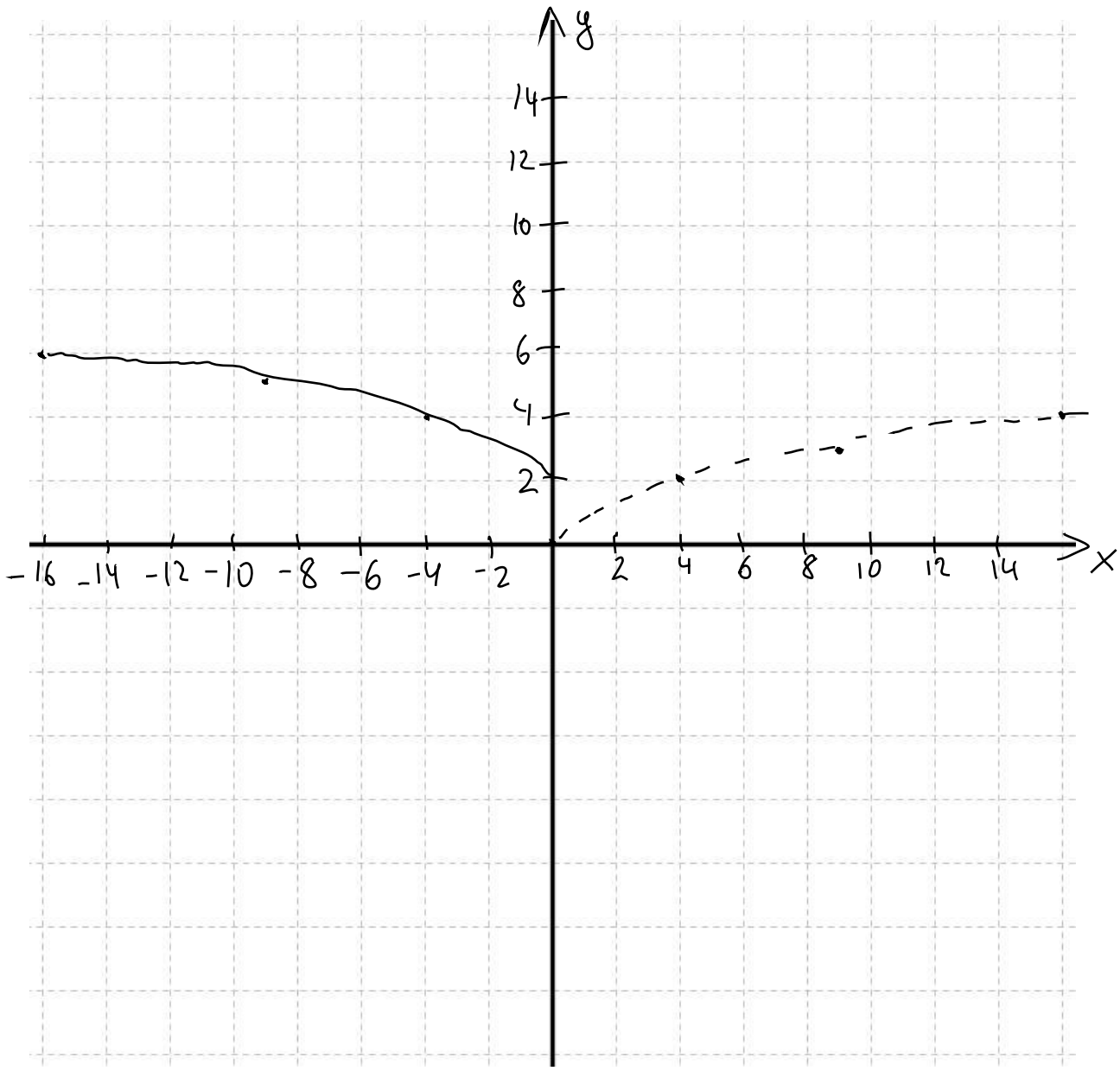
b) If f(x) is horizontally compressed by a factor of 1/2 to form j(x), then j(x) is shifted right by 5 to form k(x), what is k's equation?

$j(x) = |2x|$

$k(x) = |2(x-5)| = |2x-10|$

#6. On the axes below, graph f using a dashed line and g using a solid line.

$$f(x) = \sqrt{x}; \qquad g(x) = \sqrt{-x} + 2$$

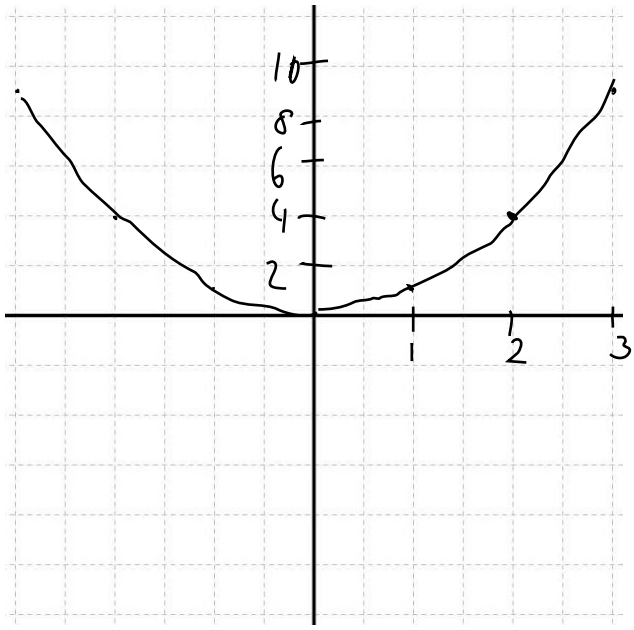


#7. Bonnie wants to graph k(x) = 3 – floor(2x). She wants to start with the graph of f(x) = floor(x), which she knows by heart, and apply a sequence of transformations to arrive at the graph of k. Which transformations should she apply, and in what order? Please number your steps.

1. horizontal compression by factor of 1/2 : g(x) = floor(2x)
2. reflection about x axis : h(x) = – floor(2x)
3. shift up by 3 : k(x) = 3 – floor(2x)

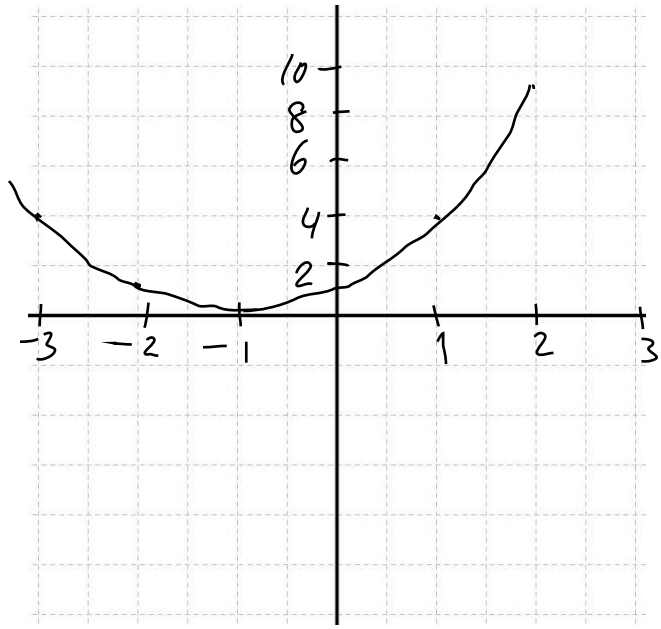
#8. Graph $k(x) = \frac{1}{2}(x+1)^2 - 3$ by applying a series of transformations to $f(x) = x^2$. Apply one transformation at a time, showing the new graph at each stage.

1. Parent: $y = x^2$



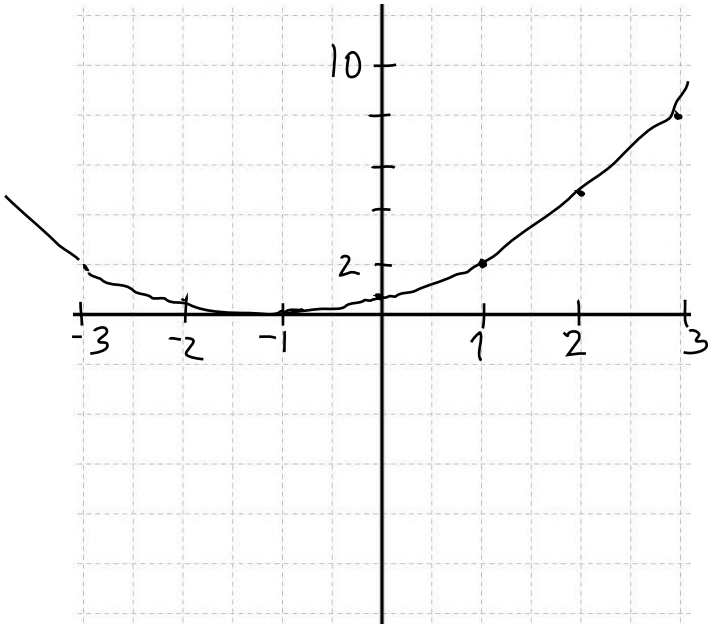
2. Transformation: shift left by 1

New equation: $y = (x+1)^2$



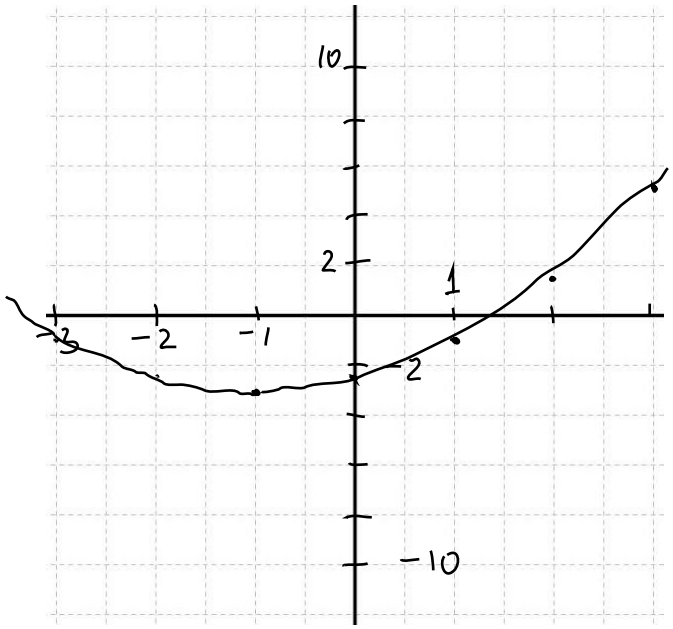
3. Transformation: vertical compression by factor of 1/2

New equation: $y = \frac{1}{2}(x+1)^2$



4. Transformation: shift down by 3

New equation: $y = \frac{1}{2}(x+1)^2 - 3$



#9. Here's a nice quadratic function: $f(x) = -2x^2 + 6x - 1$ $= -2\left(x^2 - \overset{-2\cdot\frac{3}{2}\cdot x}{3x} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right) - 1$

$= -2\underbrace{\left(x^2 - 3x + \left(\frac{3}{2}\right)^2\right)}_{\left(x-\frac{3}{2}\right)^2} + 2\cdot\left(\frac{9}{4}\right) - 1$

a) Find the coordinates of its vertex (in fraction form)

$= -2\left(x - \frac{3}{2}\right)^2 + \frac{9}{2} - 1$

$= -2\left(x - \frac{3}{2}\right)^2 + \frac{7}{2}$

vertex : $\left(\frac{3}{2}, \frac{7}{2}\right)$

b) Find the x-intercepts (if any) (in simplified radical form)

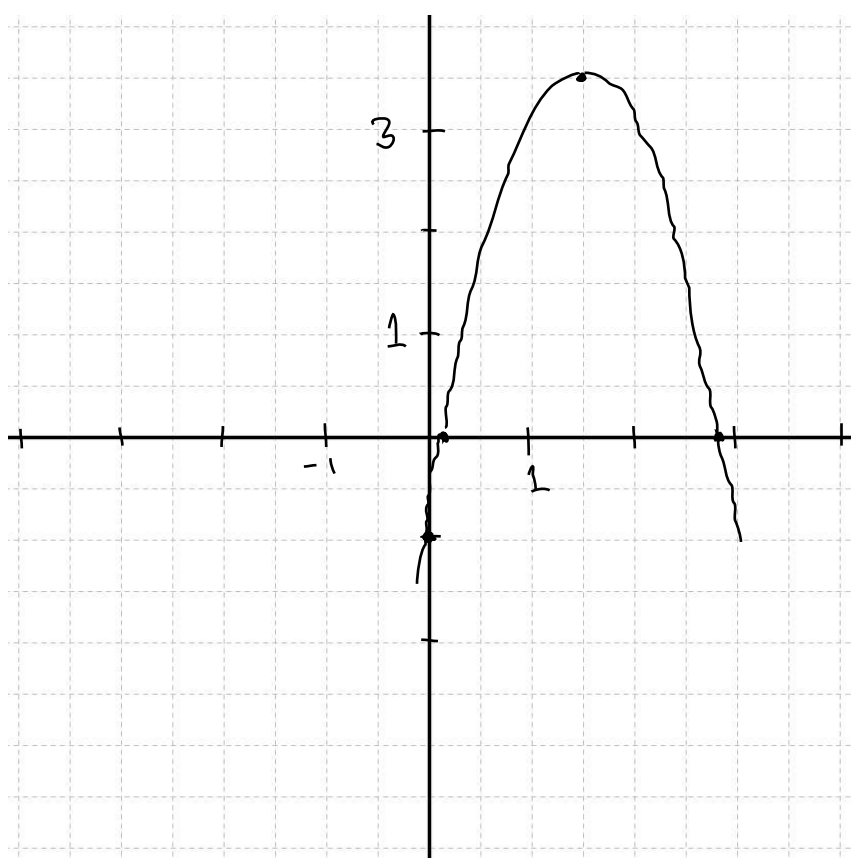
$0 = -2\left(x - \frac{3}{2}\right)^2 + \frac{7}{2}$

$\left(x - \frac{3}{2}\right)^2 = \frac{7}{4}$

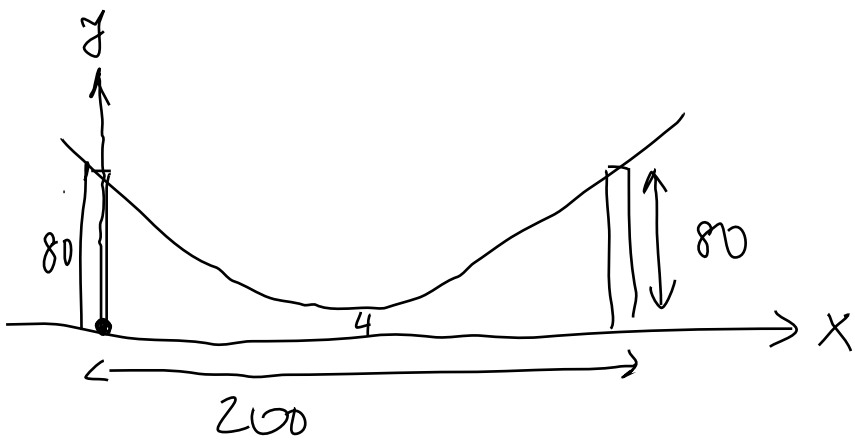
$x - \frac{3}{2} = \pm\sqrt{\frac{7}{4}} = \pm\frac{\sqrt{7}}{2} \rightarrow \boxed{x = \frac{3}{2} \pm \frac{\sqrt{7}}{2}} = 0.2 \; ; \; 2.8$

c) Find the y-intercept $f(0) = \boxed{-1}$

d) Use the above information, and the symmetry property, to sketch a graph of f.



#10. The main cable of a suspension bridge is hung from towers that are 80 feet tall (as measured from the level road surface) and 200 feet apart. At a point midway between the towers, the main cables come to a point just 4 feet above the road's surface. Draw the bridge, indicated your choice of x-y axes, then write an equation for the main cable. (Use reduced fractions not decimals)



vertex = $(100, 4)$

y-intercept = 80

$$y = a(x-100)^2 + 4$$

determine a:

$$y(0) = a(0-100)^2 + 4 = 80$$

$$10000a + 4 = 80$$

$$a = \frac{76}{10000} = \frac{19}{2500}$$

$$\boxed{y(x) = \frac{19}{2500}(x-100)^2 + 4}$$

#11. Freja sells about 200 lbs of her Swedish meatballs every week, at \$5.00/lb. Market analysis indicates that for each nickel she raises the price, sales will fall by 1.5 lb/wk. What price would maximize Freja's revenue?

$$\text{Revenue} = \underset{\substack{\uparrow \\ x}}{(\text{price})} \cdot \underset{\substack{\uparrow \\ S(x)}}{(\text{sales})}$$

x	$S(x)$
\$5.00	200 lb
\$5.05	198.5
\$5.10	197

$S(x)$ is a linear function:

$$\text{slope} = \frac{-1.5 \text{ lb}}{\$0.05} = -30$$

$$S(x) = -30x + b$$

$$S(5) = -30(5) + b = 200 \rightarrow b = 350$$

$$\boxed{S(x) = -30x + 350}$$

Revenue: $R(x) = x \cdot S(x) = x(-30x + 350) =$

$$= -30x^2 + 350x = -30\left(x^2 - \frac{350x}{30}\right) = -30\left(x^2 - (2)\frac{35}{6}x\right) =$$

$$= -30\left(x^2 - 2x \cdot \frac{35}{6} + \left(\frac{35}{6}\right)^2\right) + 30 \cdot \left(\frac{35}{6}\right)^2 =$$

$$= -30\left(x - \frac{35}{6}\right)^2 + \frac{\cancel{30}^{5} \cdot 35 \cdot 35}{\cancel{6} \cdot 6} =$$

$$= -30\left(x - \frac{35}{6}\right)^2 + \frac{6125}{6}$$

revenue max. when $x = \frac{35}{6} = \boxed{\$5.83}$ / lb

12. The flow of water over the top of a dam varies jointly with the width of the dam and the 3/2 power of the height of water over the dam. If 12000 gallons/minute flow over a 100-foot wide dam when the height of water is 32 inches, what would be the height of water over a 50-foot wide dam with 5000 gallons/minute flowing over it? Round your final answer to the nearest tenth of an inch.

$f = k w h^{3/2}$ $\quad$ $k =$ const, $w =$ width, $h =$ height

$$\begin{cases} 5000 = (k)(50)(h^{3/2}) \\ 12000 = (k)(100)(32^{3/2}) \end{cases}$$

$$\frac{5000}{12000} = \frac{\cancel{k} \cdot 50 \cdot h^{3/2}}{\cancel{k} \cdot 100 \cdot 32^{3/2}}$$

$$\frac{5}{12} = \frac{1}{2} \cdot \left(\frac{h}{32}\right)^{3/2} \rightarrow \left(\frac{h}{32}\right)^{3/2} = \frac{5}{6}$$

$$\frac{h}{32} = \left(\frac{5}{6}\right)^{2/3}$$

$$h = 32 \cdot \left(\frac{5}{6}\right)^{2/3} = \boxed{28.3 \text{ inches}}$$

#13. Given $f(x) = \frac{2x}{x-1}$ and $(x) = \frac{x-3}{x-6}$, find (f−g)(x) and state its domain.

$$(f-g)(x) = \frac{2x}{x-1} - \frac{x-3}{x-6}$$ Domain: $x \neq 1, 6$

simplify: $$\frac{2x(x-6) - (x-3)(x-1)}{(x-1)(x-6)} =$$

$$= \frac{2x^2 - 12x - x^2 + 4x - 3}{(x-1)(x-6)} =$$

$$= \frac{x^2 - 8x - 3}{(x-1)(x-6)}$$

#14. Given $f(x) = \frac{2x}{x-1}$ and $(x) = \frac{x-3}{x-6}$, find f(g(x)) and state its domain.

Domain:

$g(x)$ must be defined: $x \neq 6$

$f(g) = \frac{2g}{g-1}$ must be defined: $g \neq 1$

$g = \frac{x-3}{x-6}$ cannot equal 1 for any x

So domain of $f(g(x))$ is $\boxed{x \neq 6}$

$$f(g(x)) = \frac{2g(x)}{g(x)-1} = \frac{2\left(\frac{x-3}{x-6}\right)}{\frac{x-3}{x-6} - 1} = \frac{\frac{2(x-3)}{\cancel{x-6}}}{\frac{x-3-x+6}{\cancel{x-6}}} = \boxed{\frac{2}{3}(x-3)}$$

#15. Verify using FPI that $f(x) = \frac{1-x}{2-x}$ and $g(x) = \frac{1-2x}{1-x}$ are inverses.

$$f(g(x)) = \frac{1-g(x)}{2-g(x)} = \frac{1-\frac{1-2x}{1-x}}{2-\frac{1-2x}{1-x}} = \frac{\frac{1-x-1+2x}{\cancel{1-x}}}{\frac{2-2x-1+2x}{\cancel{1-x}}} = \frac{x}{1} = x$$

$$g(f(x)) = \frac{1-2f(x)}{1-f(x)} = \frac{1-2\cdot\frac{(1-x)}{2-x}}{1-\frac{1-x}{2-x}} = \frac{\frac{2-x-2+2x}{\cancel{2-x}}}{\frac{2-x-1+x}{\cancel{2-x}}} =$$

$$= \frac{x}{1} = x$$

That proves f and g are inverses of each other.

#16. Write an explicit description of this sequence:

12, −18, 27, −40.5, ... ← geometric sequence

ratio of -1.5

$$\boxed{a_n = 12 \cdot (-1.5)^{n-1}}$$

$a_1 = 12$, $a_2 = 12(-1.5) = -18$, $a_3 = 12(-1.5)^2 = 27 \ldots$

#17. Add up the first 800 terms of this sequence:

1, 4, 7, 10, ... ← arithmetic sequence

+3

$a_1 = 1$

$a_{800} = a_1 + (799)(3) = 1 + (799)(3) = 2398$

$$\text{sum} = \frac{(800)(a_1 + a_{800})}{2} = \frac{800\,(1 + 2398)}{2} =$$

$$= \boxed{959600}$$

#18. The population is now 120, and expected to double every 8 years. Predict the population in 12 years.

population $= y$

years after now $= x$

$y = y_0 2^{kx}$

y_0 = initial population $= 120$

Find k: $y(8) = 120 \cdot 2^{k \cdot 8} = 240 \rightarrow 2^{8k} = 2^1 \rightarrow k = \frac{1}{8}$

$y(x) = 120 \cdot 2^{x/8}$

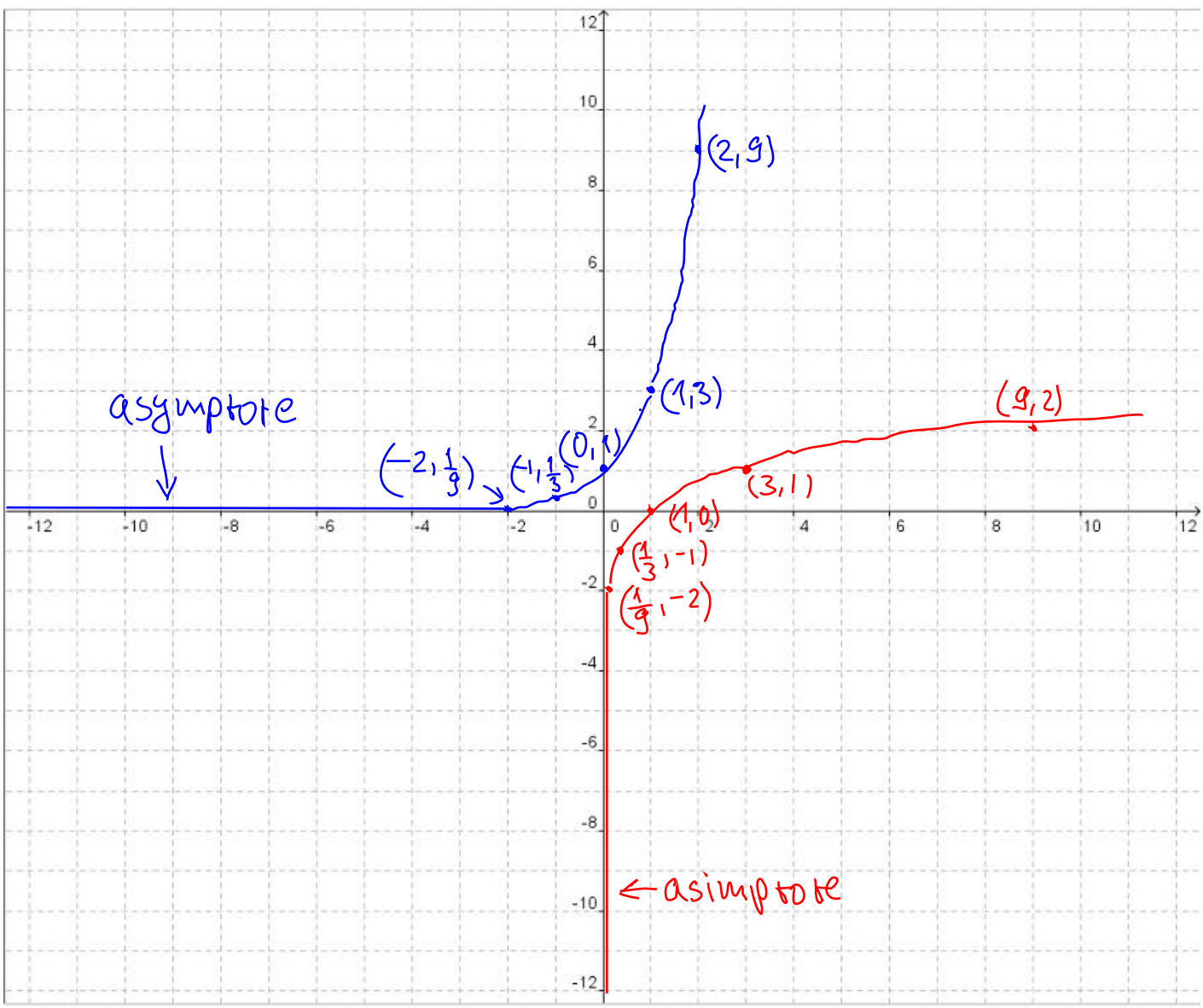
$y(12) = 120 \cdot 2^{12/8} = 120 \cdot 2^{3/2} = 339.4 = \boxed{339}$

#19. What is the inverse of $f(x) = \log_7(x)$?
Hint: Don't overthink it. This is supposed to be the easiest question on the test!

$y = \log_7 x \rightarrow x = 7^y$

the inverse is $\boxed{g(x) = 7^x}$

#20. Sketch graphs of $f(x) = 3^x$ (in blue) and $g(x) = \log_3(x)$ (in red). Label the coordinates of the five amigo points for each, AND be sure to show the asymptotes.



#21. Rewrite as a sum or difference of logarithms:

$$\log\left(\frac{x^2y}{z^3}\right) = \log x^2y - \log z^3 =$$

$$= \log x^2 + \log y - \log z^3 =$$

$$= 2\log x + \log y - 3\log z$$

#22. Rewrite as a single logarithm:

$$\log_2 x - 3\log_2(4y) + 2\log_2(x-1) =$$

$$= \log_2 x - \log_2(4y)^3 + \log_2(x-1)^2 =$$

$$= \log_2 \frac{x}{(4y)^3} + \log_2(x-1)^2 =$$

$$= \log_2\left(\frac{x(x-1)^2}{64y^3}\right)$$

#23. Solve: $12^{2x-1} = 8$ $/ \ln$

$$(2x-1)\ln 12 = \ln 8$$

$$2x-1 = \frac{\ln 8}{\ln 12} = 0.84$$

$$2x = 1.84$$

$$x = \frac{1.84}{2} = 0.92$$

#24. Solve for the inverse of f(x) = 3 + ln(2x)

$$y = 3 + \ln 2x$$

solve for x : $y - 3 = \ln 2x$

$$e^{y-3} = 2x$$

$$x = \frac{e^{y-3}}{2}$$

The inverse is: $g(x) = \dfrac{e^{x-3}}{2}$

#25. The half-life of Biffium is 7300 years. How old is a fossil the presently contains only 0.5% of its original Biffium?

x = time in years

y = amount of Biffium remaining:

$$y = y_0 \cdot 2^{-t/7300}$$

initial amount at time $t=0$

0.5%

$$0.005\,\cancel{y_0} = \cancel{y_0}\, 2^{-t/7300}$$

$$0.005 = 2^{-t/7300} \quad / \ln$$

$$\ln 0.005 = -\frac{t}{7300} \cdot \ln 2$$

$$t = -7300 \cdot \frac{\ln 0.005}{\ln 2} = \boxed{55800 \text{ years}}$$