Problem 1. (30 pts.)

Evaluate each derivative and integral.

(i) $\frac{d}{dx}\left(\arcsin(3^x)\right) = \frac{1}{\sqrt{1-(3^x)^2}} \cdot 3^x \ln 3 = \boxed{\frac{3^x \ln 3}{\sqrt{1-3^{2x}}}}$

(ii) $\frac{d}{dx}\left((3x+1)^{\sin x}\right) = \frac{d}{dx}\left(e^{\ln(3x+1)\cdot \sin x}\right) =$

$= e^{\ln(3x+1)\cdot \sin x} \cdot \frac{d}{dx}\left(\ln(3x+1)\cdot \sin x\right) =$

$= (3x+1)^{\sin x} \cdot \left(\frac{1}{3x+1} \cdot 3 \cdot \sin x + \ln(3x+1)\cdot \cos x\right) =$

$= \boxed{(3x+1)^{\sin x} \cdot \left(\frac{3\sin x}{3x+1} + \ln(3x+1)\cdot \cos x\right)}$

(iii) $\int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{1+\sin^2 x} =$

$u = \sin x$

$du = \cos x\, dx$

$\sin(0) = 0$

$\sin(\frac{\pi}{2}) = 1$

$= \int_0^1 \frac{du}{1+u^2} = \arctan u \Big|_0^1 = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$

Problem 2. (50 pts.)

Evaluate each integral.

By parts: $u = \arctan x$

$dv = dx \to v = x$

(i) $\displaystyle\int \arctan x \, dx = x \cdot \arctan x - \int x \cdot \frac{1}{1+x^2} \, dx =$

$$= x \cdot \arctan x - \frac{1}{2} \cdot \int \frac{d(1+x^2)}{1+x^2}$$

$$= \boxed{x \cdot \arctan x - \frac{1}{2} \cdot \ln(1+x^2) + C}$$

(i) $\displaystyle\int \frac{\cos^3 x}{\sin^2 x} dx = \int \frac{\cos^2 x}{\sin^2 x} \cdot \cos x \, dx =$

$u = \sin x$

$du = \cos x \, dx$

$$= \int \frac{1-\sin^2 x}{\sin^2 x} \cdot \cos x \, dx =$$

$$= \int \frac{1-u^2}{u^2} du = \int \frac{du}{u^2} - \int du = -\frac{1}{u} - u + C =$$

$$= \boxed{-\csc x - \sin x + C}$$

(ii) $\displaystyle\int_0^{\frac{\pi}{4}} \sec x \tan^5 x \, dx = \int_0^{\pi/4} (\underbrace{\tan^2 x}_{\sec^2 x - 1})^2 \cdot \tan x \cdot \sec x \, dx =$

$u = \sec x$

$du = \tan x \cdot \sec x \, dx$

$\sec(0) = 1$

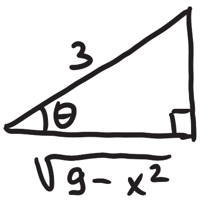
$\sec(\pi/4) = \sqrt{2}$

$$= \int_1^{\sqrt{2}} (u^2-1)^2 \, du = \int_1^{\sqrt{2}} (u^4 - 2u^2 + 1) \, du =$$

$$= \frac{u^5}{5}\Big|_1^{\sqrt{2}} - \frac{2u^3}{3}\Big|_1^{\sqrt{2}} + u\Big|_1^{\sqrt{2}} = \frac{1}{5}(4\sqrt{2} - 1) - \frac{2}{3}(2\sqrt{2} - 1) + \sqrt{2} - 1 =$$

$$= \underbrace{\sqrt{2}\left(\frac{4}{5} - \frac{4}{3} + 1\right)}_{\frac{12-20+15}{15}} \underbrace{- \frac{1}{5} + \frac{2}{3} - 1}_{\frac{-3+10-15}{15}} = \boxed{\frac{7}{15} \cdot \sqrt{2} - \frac{8}{15}}$$

(i) $\int\sqrt{9-x^2}\,dx \underset{\substack{\uparrow \\ x=3\sin\theta}}{=} \int \underbrace{\sqrt{9-9\sin^2\theta}}_{3\cos\theta}\cdot 3\cos\theta\, d\theta =$



$dx = 3\cos\theta\, dx$

$= 9\int \cos^2\theta\, d\theta = 9\int \frac{1+\cos(2\theta)}{2}\cdot d\theta =$

$2\sin\theta\cdot\cos\theta$

$= \frac{9}{2}\int d\theta + \frac{9}{4}\int \cos(2\theta)\, d(2\theta) = \frac{9}{2}\cdot\theta + \frac{9}{4}\sin(2\theta) + C =$

$= \frac{9}{2}\cdot \arcsin\left(\frac{x}{3}\right) + \frac{9}{4}\cdot 2\cdot \underset{\substack{\uparrow \\ \sin\theta}}{\frac{x}{3}} \cdot \underset{\substack{\uparrow \\ \cos\theta}}{\frac{\sqrt{9-x^2}}{3}} + C = \boxed{\frac{9}{2}\cdot \arcsin\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C}$

(ii) $\int \frac{1}{x^2(x+1)}\,dx$

$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \rightarrow 1 = Ax(x+1) + B(x+1) + Cx^2$

$1 = x^2\underbrace{(A+C)}_{0} + x\underbrace{(A+B)}_{0} + \underset{\substack{\uparrow \\ 1}}{B}$

$\Rightarrow B = 1,\ A = -B = -1,\ C = -A = 1$

$\Rightarrow \int \frac{1}{x^2(x+1)}\,dx = \int\left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right)dx =$

$= -\int \frac{dx}{x} + \int x^{-2}\,dx + \int \frac{dx}{x+1} =$

$= \boxed{-\ln|x| - \frac{1}{x} + \ln|x+1| + C}$

Problem 3. (20 pts.)

Determine if the integral is convergent or divergent. Evaluate those that are convergent.

(i) $\int_0^2 \frac{1}{\sqrt{4-x^2}}dx = \int_0^2 \frac{d(x/2)}{\sqrt{1-\left(\frac{x}{2}\right)^2}} = \arcsin\left(\frac{x}{2}\right)\Big|_0^2 =$

undefined at $x = 2$

$= \lim_{x\to 2} \arcsin\left(\frac{x}{2}\right) - \arcsin\left(\frac{0}{2}\right) =$

$= \arcsin(1) - \arcsin(0) = \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$

$\boxed{\text{Convergent}}$

(ii) $\int_{-\infty}^{\infty} \frac{2x}{x^2+1}dx = \int_{-\infty}^{0} \frac{2x\,dx}{x^2+1} + \int_{0}^{\infty} \frac{2x\,dx}{x^2+1} =$

$= \ln|x^2+1|\Big|_{-\infty}^{0} + \ln|x^2+1|\Big|_0^{\infty} =$

$= \ln 1 - \underbrace{\lim_{x\to-\infty} \ln|x^2+1|}_{\infty} + \underbrace{\lim_{x\to\infty} \ln|x^2+1|}_{\infty} - \ln 1$

The integral is $\boxed{\text{divergent}}$.

Problem 4. (20 pts.)

Find the sum of the series.

(i) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ $= \sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{n+2-n}{n(n+2)} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \left[\frac{n+2}{n(n+2)} - \frac{n}{n(n+2)} \right] =$

$= \frac{1}{2} \cdot \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+2} \right] = \frac{1}{2} \cdot \left[\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \dots \right]$

cancel out (telescopic series)

$= \frac{1}{2} \cdot \left[\frac{1}{1} + \frac{1}{2} \right] = \frac{1}{2} \cdot \frac{3}{2} = \boxed{\frac{3}{4}}$

(ii) $\sum_{n=2}^{\infty} \frac{5}{2^{n+2}}$ $= \frac{5}{2^4} + \frac{5}{2^5} + \frac{5}{2^6} + \frac{5}{2^7} + \dots =$

$= \frac{5}{2^4} \left(\underbrace{1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots}_{\text{geometric series sum } = \frac{1}{1-\frac{1}{2}}} \right) =$

$= \frac{5}{2^4} \cdot \frac{1}{1-\frac{1}{2}} = \frac{5}{2^4} \cdot 2 = \frac{5}{2^3} = \boxed{\frac{5}{8}}$

Problem 5. (40 pts.)

(a) By using an appropriate test (specify the name) determine if each series is convergent or divergent.

(i) $\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$

(ii) $\sum_{n=1}^{\infty} \frac{n^{n+1}}{n!}$

(i) convergent by comparison test:

$$0 \leq \sum \frac{n^2}{n^4+1} \leq \sum \frac{n^2}{n^4} = \sum \frac{1}{n^2}$$

convergent p-series, $p = 2$

(ii) divergent by the limit test:

$$\frac{n^{n+1}}{n!} = \frac{(n)(n)(n)\ldots(n)(n)}{(1)(2)(3)\ldots(n)} \geq (1)(1)(1)\ldots(1)(n) = n$$

diverges as $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} \frac{n^{n+1}}{n!} = \infty$$

(b) Determine if the series converges absolutely, converges conditionally, or diverge.

(i) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^4+1}}$

(ii) $\sum_{n=1}^{\infty} (-1)^n \arctan(n)$

(i) converges absolutely by comparison test:

$$0 \leq \sum \left| \frac{(-1)^n}{\sqrt{n^4+1}} \right| \leq \sum \frac{1}{\sqrt{n^4}} = \sum \frac{1}{n^2}$$

smaller denominator

convergent p-series, $p = 2$

(ii) diverges by the limit test:

$$\lim_{n\to\infty} \arctan(n) = \pi/2$$

does not go to zero

$$\Rightarrow \lim_{n\to\infty} (-1)^n \arctan(n) \text{ does not exist.}$$

sign alternates

Problem 6. (30 pts.)

Find the radius of convergence and interval of convergence of each series.

(i) $\sum_{n=1}^{\infty} \frac{n!(x-1)^n}{2^n}$

Ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!\,|x-1|^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n!\,|x-1|^n} =$

$= \lim_{n\to\infty} \frac{(n+1)|x-1|}{2} = \begin{cases} 0 & \text{if } x=1 \\ \infty & \text{if } x\neq 1 \end{cases}$

=> Interval of convergence is just the point $\boxed{x=1}$

Radius of convergence $= \boxed{0}$

(ii) $\sum_{n=1}^{\infty} \frac{n(x-1)^n}{2^n}$

Ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)|x-1|^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n|x-1|^n} =$

$= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{|x-1|}{2} = \frac{|x-1|}{2} < 1$

$\boxed{\text{Radius} = 2}$

=> Absolutely convergent when $|x-1| < 2 \to -1 < x < 3$

Check $x=-1$:

$\sum_{n=1}^{\infty} \frac{n(-2)^n}{2^n} = \sum_{n=1}^{\infty} n(-1)^n$: divergent since $\lim_{n\to\infty} n(-1)^n \neq 0$

Check $x=3$:

$\sum_{n=1}^{\infty} \frac{n(2)^n}{2^n} = \sum_{n=1}^{\infty} n$: divergent since $\lim_{n\to\infty} n \neq 0$

=> Interval of convergence is $\boxed{(-1, 3)}$

Problem 7. (30 pts.)

Find the power series by using a known series or Taylor expansion for each function centered at 0 with at least 3 nonzero terms.

(i) $f(x)=\dfrac{e^{2x}-1}{x}$ centered at x=0

(ii) $f(x)=x^{\frac{3}{7}}$ centered at x=1

(i) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$

$$f(x)=\frac{e^{2x}-1}{x}=\frac{\cancel{1}+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\frac{(2x)^4}{4!}+\cdots-\cancel{1}}{x}=$$

$$=\frac{2x}{x}+\frac{(2x)^2}{2!\,x}+\frac{(2x)^3}{3!\,x}+\frac{(2x)^4}{4!\,x}+\cdots=2+\frac{4x^2}{2x}+\frac{8x^3}{6x}+\frac{16x^4}{24x}+\cdots$$

$$=\boxed{2+2x+\frac{4}{3}x^2+\frac{2}{3}x^3+\cdots}$$

(ii)

$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}\cdot x^2+\cdots$ (Binomial Series)

$$f(x)=x^{3/7}=(1+\underbrace{(x-1)}_{x\to(x-1)})^{3/7}=1+\left(\tfrac{3}{7}\right)(x-1)+\frac{\left(\tfrac{3}{7}\right)\left(\tfrac{3}{7}-1\right)(x-1)^2}{2!}+\cdots$$

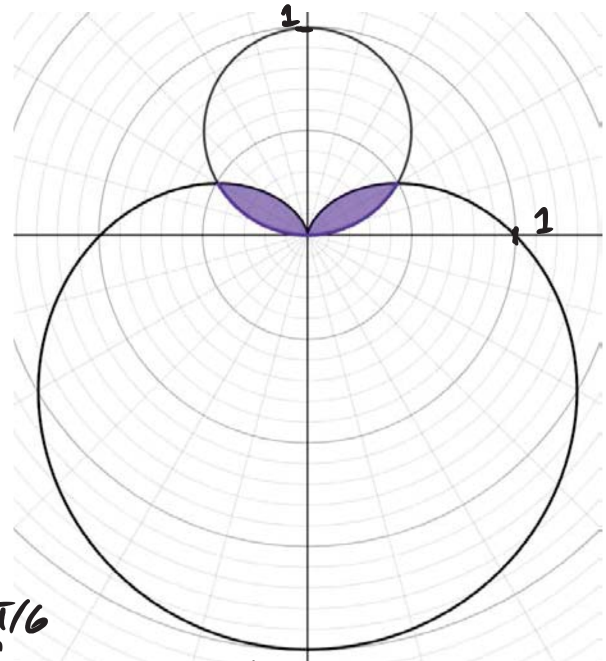
$\leftarrow n=3/7$

$$=1+\frac{3}{7}(x-1)+\frac{3}{7}\cdot\left(-\frac{4}{7}\right)\cdot\frac{(x-1)^2}{2}+\cdots$$

$$=\boxed{1+\frac{3}{7}(x-1)-\frac{6}{49}(x-1)^2+\cdots}$$

Problem 8. (30 pts.)

Find the area between $r = 1 - \sin\theta$ and the circle $r = \sin\theta$ indicated in the picture.



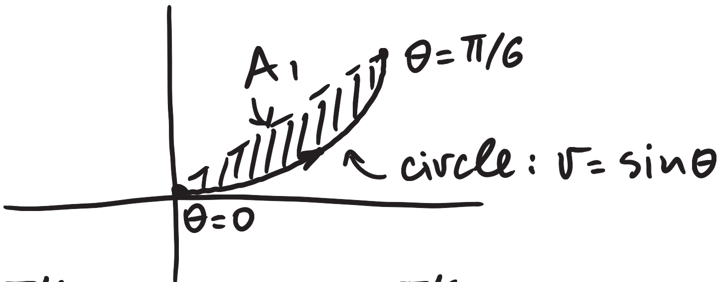
intersection points:

$r = \sin\theta = 1 - \sin\theta$

$2\sin\theta = 1$

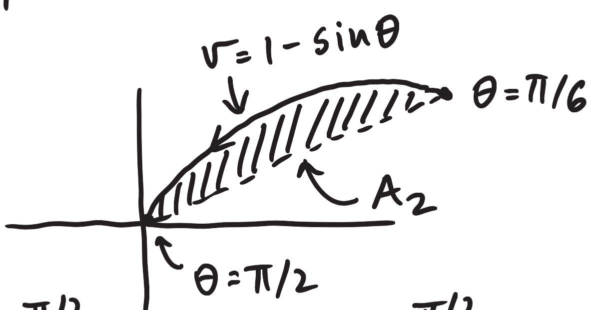
$\sin\theta = 1/2 \to \theta = \pi/6$

$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$



$$A_1 = \int_0^{\pi/6} \frac{1}{2} r^2(\theta)\,d\theta = \int_0^{\pi/6} \frac{1}{2}\sin^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/6} \frac{1-\cos(2\theta)}{2}\cdot d\theta =$$

$$= \frac{1}{4}\theta\Big|_0^{\pi/6} - \frac{1}{8}\sin(2\theta)\Big|_0^{\pi/6} = \frac{1}{4}\left(\frac{\pi}{6}\right) - \frac{1}{8}\frac{\sqrt{3}}{2} = \frac{\pi}{24} - \frac{\sqrt{3}}{16}$$



$$A_2 = \int_{\pi/6}^{\pi/2} \frac{1}{2} r^2(\theta)\,d\theta = \int_{\pi/6}^{\pi/2} \frac{1}{2}(1-\sin\theta)^2\,d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/2}(1 - 2\sin\theta + \sin^2\theta)\,d\theta =$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/2}\left(1 - 2\sin\theta + \frac{1-\cos(2\theta)}{2}\right)d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/2}\left(\frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos(2\theta)\right)d\theta =$$

$$= \left[\frac{3\theta}{4} + \cos\theta - \frac{1}{8}\sin(2\theta)\right]\Bigg|_{\pi/6}^{\pi/2} = \frac{3}{4}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + \left(0 - \frac{\sqrt{3}}{2}\right) - \frac{1}{8}\left(0 - \frac{\sqrt{3}}{2}\right) =$$

$$= \frac{\pi}{4} - \frac{7}{16}\sqrt{3}$$

Total area $A = 2(A_1 + A_2) = 2\left(\frac{\pi}{24} - \frac{\sqrt{3}}{16} + \frac{\pi}{4} - \frac{7}{16}\sqrt{3}\right) = \boxed{\frac{7\pi}{12} - \sqrt{3}}$

Problem 9. (20 pts.)

(a) Find two unit vectors orthogonal to both $\vec{v} = \langle 1, -2, 4 \rangle$ and $\vec{u} = \langle -5, 2, 3 \rangle$

(b) Find the area of a parallelogram with two adjacent sides $\vec{v}$ and $\vec{u}$.

(a) $\vec{v} \times \vec{u}$ is orthogonal to both $\vec{v}$ and $\vec{u}$:

$$\vec{v} \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ -5 & 2 & 3 \end{vmatrix} = \underbrace{\begin{vmatrix} -2 & 4 \\ 2 & 3 \end{vmatrix}}_{-14} \hat{i} - \underbrace{\begin{vmatrix} 1 & 4 \\ -5 & 3 \end{vmatrix}}_{23} \hat{j} + \underbrace{\begin{vmatrix} 1 & -2 \\ -5 & 2 \end{vmatrix}}_{-8} \hat{k} =$$

$$= \langle -14, -23, -8 \rangle$$

$$|\vec{v} \times \vec{u}| = \sqrt{(-14)^2 + (-23)^2 + (-8)^2} = \sqrt{789}$$

first unit vector:

$$\hat{u}_1 = \frac{\vec{v} \times \vec{u}}{|\vec{v} \times \vec{u}|} = \boxed{\left\langle \frac{-14}{\sqrt{789}}, \frac{-23}{\sqrt{789}}, \frac{-8}{\sqrt{789}} \right\rangle}$$

second unit vector:

$$\hat{u}_2 = -\hat{u}_1 = \boxed{\left\langle \frac{14}{\sqrt{789}}, \frac{23}{\sqrt{789}}, \frac{8}{\sqrt{789}} \right\rangle}$$

(ii)

$$\text{Area} = |\vec{v} \times \vec{u}| = \boxed{\sqrt{789}}$$

Problem 10. (30 pts.)

Find the length of the following curve on the given intervals.

$r(t) = \langle e^{3t}+1, e^{3t}-1, 3e^{3t} \rangle$; $0 \le t \le \ln 2$

$x(t) = e^{3t}+1 \rightarrow \frac{dx}{dt} = 3e^{3t}$

$y(t) = e^{3t}-1 \rightarrow \frac{dy}{dt} = 3e^{3t}$

$z(t) = 3e^{3t} \rightarrow \frac{dz}{dt} = 9e^{3t}$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 9e^{6t} + 9e^{6t} + 81e^{6t} = 99e^{6t}$

$\text{Length} = \int_0^{\ln 2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \cdot dt =$

$= \int_0^{\ln 2} \sqrt{99e^{6t}}\, dt = \sqrt{9} \cdot \sqrt{11} \cdot \int_0^{\ln 2} e^{3t}\, dt =$

$= \cancel{3}\sqrt{11} \cdot \frac{e^{3t}}{\cancel{3}} \Big|_0^{\ln 2} = \sqrt{11}\left(e^{3\ln 2} - e^{0}\right) =$

$= \sqrt{11}\left(\left(e^{\ln 2}\right)^3 - 1\right) = \sqrt{11}\left(2^3 - 1\right) = \boxed{7\sqrt{11}}$