	FINAL EXAM	
	10 pages - 15 problems	
200 points		**NAME**

DIRECTIONS: Show all the work in the space provided. Box final answers, and follow the indicated directions. Problem #3 is worth 20 pts, Problem #6 is worth 25 points. All other problems are worth 12pts each.

1) Find the value of *c* so that the function *f* is continuous on the entire real line.

$$f(x)=\begin{cases} x^2+3, x\le 1 \\ cx+4, x>1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2+3) = 1+3=4 = f(1)$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (cx+4) = c+4$$

$$4 = c+4 \quad \to \quad \boxed{c=0}$$

2) Let $f(x)=x^2-x$. Calculate by definition $f'(3)$

$$f(3) = 3^2-3 = 6$$

$$f(3+h) = (3+h)^2-(3+h) = h^2+5h+6$$

$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(h)}{h} =$$

$$= \lim_{h\to 0} \frac{h^2+5h+\cancel{6}-\cancel{6}}{h} =$$

$$= \lim_{h\to 0} (h+5) = 0+5 = \boxed{5}$$

3) Calculate:

a) $\lim_{x\to 9}\frac{\sqrt{x}-3}{x^2-81}$. $= \lim_{x\to 9} \frac{\sqrt{x}-3}{(x-9)(x+9)} =$

$$= \lim_{x\to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)(x+9)} =$$

$$= \lim_{x\to 9} \frac{1}{(\sqrt{x}+3)(x+9)} = \frac{1}{(3+3)(9+9)} = \boxed{\frac{1}{108}}$$

b) $\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}$. $=$

$$= \underbrace{\lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta}}_{1} \cdot 2 = (1)(2) = \boxed{2}$$

c) $\lim_{\theta\to\infty}\frac{\sin 2\theta}{\theta}$. $= 0$ by Squeeze Theorem:

$$-\frac{1}{\theta} \le \frac{\sin 2\theta}{\theta} \le \frac{1}{\theta}$$

$$\to 0 \leftarrow$$

d) $\lim_{x\to-\infty}\frac{\sqrt{x^2+x}-3}{-2x}$ $=$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2+x}}{-2x} + \lim_{x\to-\infty} \frac{3}{2x} =$$

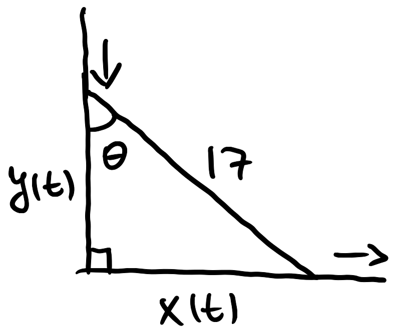
$$= \lim_{x\to-\infty} \frac{\sqrt{x^2+x}}{+2\sqrt{x^2}} + 0 =$$

$x = -|x| = -\sqrt{x^2}$, when $x<0$

$$= \lim_{x\to-\infty} \frac{1}{2}\sqrt{\frac{x^2+x}{x^2}} = \frac{1}{2}\lim_{x\to-\infty}\sqrt{1+\frac{1}{x}} = \frac{1}{2}\sqrt{1+0} = \boxed{\frac{1}{2}}$$

4) A ladder 17 feet long is leaning against a vertical wall. The top of the ladder is sliding down the wall at a rate of 2 feet per second.

(a) Draw and label the diagram for this application problem. How fast is the foot of the ladder moving away from the wall when the foot is 8 feet from the base of the wall?



$$x(t)^2 + y(t)^2 = 17^2 \quad / \frac{d}{dt}(\dots)$$

$$2xx' + 2yy' = 0$$

$x = 8\text{ ft}$

$y = \sqrt{17^2 - 8^2} = 15\text{ ft}$

$y' = -2\text{ ft/sec}$

$$x' = -\frac{y}{x} \cdot y' = -\frac{15}{8} \cdot (-2\text{ ft/sec}) = \frac{15}{4} = \boxed{3.75\text{ ft/sec}}$$

(b) Find the rate at which the angle between the ladder and the wall is changing when the foot of the ladder is 8 feet from the base of the wall.

$$\cos\theta(t) = \frac{y(t)}{17} \quad / \frac{d}{dt}(\dots)$$

$$-\sin\theta \cdot \theta' = \frac{y'}{17}$$

$$\theta' = -\frac{y'}{17\sin\theta} = -\frac{(-2\text{ ft/sec})}{(17\text{ ft})\left(\frac{8}{17}\right)} = \frac{1}{4} = \boxed{0.25\ \frac{\text{rad}}{\text{sec}}}$$

5) Consider the function f defined implicitly by:
$x + y - 1 = \ln(x^2 + y^2)$.

a) Find and sketch the tangent line to f at the point (1,0).

$$x + y - 1 = \ln(x^2 + y^2) \quad / \frac{d}{dx}(\dots)$$

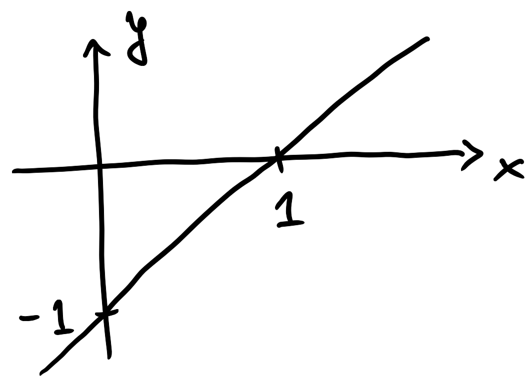
$$1 + y' - 0 = \frac{2x + 2yy'}{x^2 + y^2} \quad / \; x = 1,\; y = 0$$

$$1 + y' = \frac{2+0}{1+0} \rightarrow y' = 2 - 1 = 1$$

Tangent of slope $m = 1$, through $(1,0)$:

$$y - 0 = 1(x - 1)$$

$$\boxed{y = x - 1}$$



b) Approximate *f(1.1)* using the tangent line approximation from part a).

$$f(x) \approx x - 1$$

$$f(1.1) \approx 1.1 - 1 = \boxed{0.1}$$

6) Differentiate:

a) $f(x) = \left(\frac{x-3}{x^2+1}\right)^2$

$$f'(x) = 2\frac{(x-3)}{x^2+1} \cdot \frac{x^2+1-(x-3)2x}{(x^2+1)^2} = \frac{2(x-3)(-x^2+6x+1)}{(x^2+1)^3}$$

b) $g(x) = \sin(x^2)$

$$g'(x) = \cos(x^2) \cdot 2x = 2x\cos(x^2)$$

c) $h(x) = \sqrt{e^{2x} + e^{-2x}}$

$$h'(x) = \frac{1}{2\sqrt{e^{2x}+e^{-2x}}} \cdot (2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x}+e^{-2x}}}$$

d) $F(x) = \int_x^{x^2} \frac{\sin t}{t}\, dt$

$$F'(x) = \frac{\sin(x^2)}{x^2} \cdot 2x - \frac{\sin(x)}{x} \cdot 1 = \frac{2\sin(x^2) - \sin x}{x}$$

e) $k(x) = x^x$

$$\ln k(x) = x \ln x$$

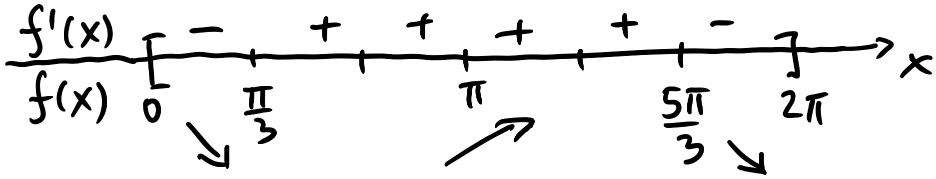
$$\frac{k'}{k} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$k' = (\ln x + 1)\, k(x) = \boxed{(\ln x + 1)\, x^x}$$

7) Find the relative extrema of the function $f(x)=\frac{1}{2}x-\sin x$ in the interval $[0,2\pi]$.

$f'(x)=\frac{1}{2}-\cos x=0 \rightarrow \cos x=\frac{1}{2}$

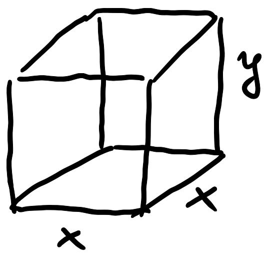
$x=\frac{\pi}{3}$ or $x=\frac{5\pi}{3}$



Rel. min: $y\left(\frac{\pi}{3}\right)=\frac{\pi}{6}-\frac{\sqrt{3}}{2}$

Rel. max: $y\left(\frac{5\pi}{3}\right)=\frac{5\pi}{6}+\frac{\sqrt{3}}{2}$

8) A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximal volume?



Area = base + sides (no top) =

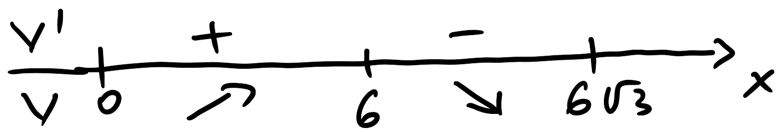
$= x^2+4xy=108$

$y=\frac{108-x^2}{4x}=\frac{27}{x}-\frac{x}{4}$

$\left.\begin{array}{l} x>0 \\ y=\frac{108-x^2}{4x}>0 \rightarrow x<\sqrt{108} \end{array}\right\} \quad 0<x<6\sqrt{3}$

Volume: $V=x^2\cdot y=27x-\frac{x^3}{4}$

$V'=27-\frac{3}{4}x^2=0 \rightarrow x=6$ or $\cancel{x=-6}$



Maximal volume at $x=6$ inches,

$y=\frac{27}{6}-\frac{6}{4}=3$ inches $\boxed{\text{dimensions } 6\times 6\times 3}$

9) Analyze the function $f(x) = x^2 e^{-x}$. Report extrema, inflection points, intervals of monotonicity and concavity, and asymptotes. Graph the function by hand based on your analysis.

Domain: all x, no V.A.

$\lim_{x\to\infty} \frac{x^2}{e^x} = 0$, $\lim_{x\to -\infty} \frac{x^2}{e^x} = \infty$, H.A.: $y = 0$

No oblique.

$f(0) = 0 \rightarrow$ y-intercept $(0,0)$

$0 = \frac{x^2}{e^x} \rightarrow x = 0 \rightarrow$ x-intercept $(0,0)$

$f(x) > 0$ for $x \neq 0$

$f(x)$: $+ \; + \; +$ | 0 | $+ \; + \; +$ $\rightarrow x$

$f'(x) = 2xe^{-x} - x^2 e^{-x} = \frac{x(2-x)}{e^x} = 0$

$x = 0$ or $x = 2$

$f'(x)$: $- \; -$ | 0 | $+ \; +$ | 2 | $- \; -$ $\rightarrow x$

$f(x)$: $\searrow$ | 0 | $\nearrow$ | 2 | $\searrow$

$y_{min} = 0$ at $x = 0$

$y_{max} = \frac{4}{e^2} \approx 0.54$ at $x = 2$

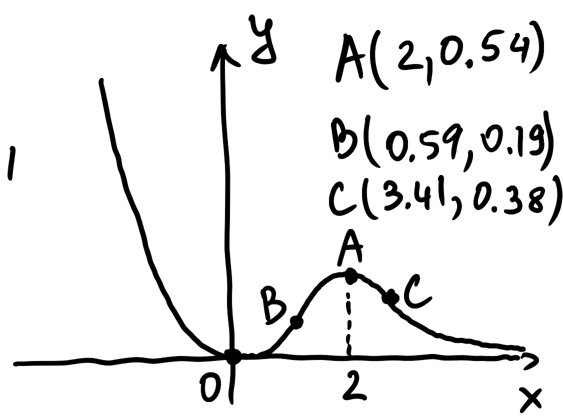
$f'' = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2 e^{-x} =$

$= \frac{x^2 - 4x + 2}{e^x} = 0$

$x = 2 - \sqrt{2} \approx 0.59$ or $x = 2 + \sqrt{2} \approx 3.41$

$f''(x)$: $+ \; +$ | $2-\sqrt{2}$ | $- \; - \; -$ | $2+\sqrt{2}$ | $+ \; + \; +$ $\rightarrow x$

$f(x)$: $\cup$ | $2-\sqrt{2}$ | $\cap$ | $2+\sqrt{2}$ | $\cup$



10) Find $\left(f^{-1}\right)'(14)$ if $f(x) = x\sqrt{x-3}$.

Let $g(x) = f^{-1}(x)$

$x\sqrt{x-3} = 14 \to x = 7 \to g(14) = 7$

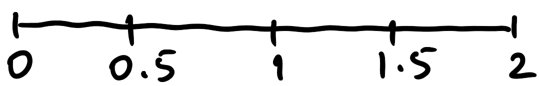
$$f'(x) = \sqrt{x-3} + \frac{x}{2\sqrt{x-3}}$$

$$f'(7) = \sqrt{4} + \frac{7}{2\sqrt{4}} = 2 + \frac{7}{4} = \frac{15}{4}$$

$$g'(14) = \frac{1}{f'(g(14))} = \frac{1}{f'(7)} = \frac{1}{\frac{15}{4}} = \boxed{\frac{4}{15}}$$

11) a) Approximate $\int_0^2 \frac{10}{x^2+1}dx$ using left Riemann sum with 4 equal subintervals.

$$\Delta x = \frac{2}{4} = \frac{1}{2}$$

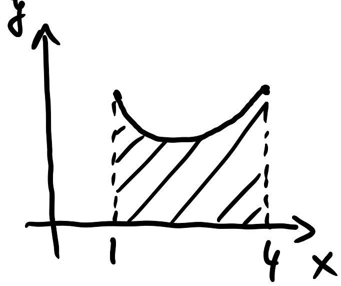


$$RS = \Delta x\,(f(0) + f(0.5) + f(1) + f(1.5)) =$$
$$= \frac{1}{2}\cdot\left(10 + 8 + 5 + \frac{40}{13}\right) = \frac{339}{26} \approx \boxed{13.038}$$

b) Calculate the exact value of the integral using Fundamental Theorem of Calculus.

$$\int_0^2 \frac{10\,dx}{x^2+1} = 10\arctan x \Big|_0^2 =$$
$$= 10\arctan 2 \approx \boxed{11.071}$$

12) Find the area of the region bounded by the curve $y = \dfrac{x^2+4}{x}$, and lines $x=1$, $x=4$, $y=0$.



$$\text{Area} = \int_1^4 \frac{x^2+4}{x}\,dx =$$

$$= \int_1^4 (x + 4x^{-1})\,dx =$$

$$= \frac{x^2}{2}\Big|_1^4 + 4\ln x\Big|_1^4 =$$

$$= \frac{1}{2}(16-1) + 4(\ln 4 - 0) =$$

$$= \frac{15}{2} + 4\ln 4 \approx 13.045$$

13) Calculate: $\displaystyle\int_3^6 \frac{x}{3\sqrt{x^2-8}}\,dx$ =

$$\left[\begin{array}{l} u = x^2 - 8 \\ du = 2x\,dx \rightarrow x\,dx = \frac{1}{2}\cdot du \\ x = 3 \rightarrow u = 9-8 = 1 \;;\quad x = 6 \rightarrow u = 36-8 = 28 \end{array}\right]$$

$$= \int_1^{28} \frac{\frac{1}{2}du}{3\sqrt{u}} = \frac{1}{6}\int_1^{28} u^{-1/2}\,du =$$

$$= \frac{1}{6}\,\frac{u^{1/2}}{\frac{1}{2}}\Big|_1^{28} = \frac{1}{3}\left(\sqrt{28} - \sqrt{1}\right) =$$

$$= \frac{1}{3}\left(2\sqrt{7} - 1\right) \approx 1.431$$

14) Find $\int \frac{e^{\frac{1}{x}}}{x^2}\, dx$. $=$

$$\left[\begin{array}{l} u = \frac{1}{x} \\ du = -\frac{1}{x^2}\,dx \rightarrow \frac{dx}{x^2} = -du \end{array}\right]$$

$$= \int e^u \cdot (-du) = -e^u + C =$$

$$= \boxed{-e^{\frac{1}{x}} + C}$$

15) Find $\int \frac{dx}{x^2 - 4x + 7}$. $= \int \frac{dx}{x^2 - 4x + 4 + 3} = \int \frac{dx}{(x-2)^2 + (\sqrt{3})^2} =$

$$\left[\begin{array}{l} u = x-2 \\ du = dx \end{array}\right]$$

$$= \int \frac{du}{(\sqrt{3})^2 + u^2} = \frac{1}{\sqrt{3}} \arctan \frac{u}{\sqrt{3}} + C =$$

$$= \boxed{\frac{1}{\sqrt{3}} \arctan\left(\frac{x-2}{\sqrt{3}}\right) + C}$$